Here G_m is the shear modulus; k_m is the limiting shear stress for this part of the curve (the flow limit). We will consider the inclusion material (high strength aluminum oxide particles with large moduli) as ideally elastic throughout the deformation process: μ_f = const. Equations (2.4)-(2.6) were solved numerically by computer using the method of successive approximations.

Figure 1 displays a comparison of theoretical and experimental load-extension curves for an SAP composite (14% Al_2O_3). The experimental results, taken from [8], are shown as points in Fig. 1. The computed values from formulas (2.4)-(2.6) are shown as solid lines. The calculated quantities are: E_m = 71 GPa; E_f = 2500 GPa; ν_m = 0.34; ν_f = 0.2; k_m = 25 MPa; c_f = 0.14.

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STABILITY OF A VISCOELASTIC ROD WITH A SPORADIC LONGITUDINAL LOAD

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UDC 539.3

The stability in an infinite time interval is studied for a viscoelastic rod compressed by a sporadic force. Rod bending is considered in a dynamic arrangement. Stability conditions are formulated in a root-mean-square for a viscoelastic rod with an arbitrary form of degree of stress relaxation and different types of end fastening. It is shown that with fulfillment of the conditions obtained a viscoelastic rod is stable, but a corresponding elastic rod with a long-term elasticity modulus is unstable.

Questions of stability for a rod made of aging viscoelastic material with an arbitrary relaxation nucleus were considered in [1, 2]. The problem was studied in a quasistatic arrangement with a deterministic compressive load. A review of studies of the stability of viscoelastic structural elements is contained in [3]. Stability conditions for elastic bodies with a sporadic load are given in [4]. The stability elastic and viscoelastic rods with a sporadic longitudinal load is analyzed in [5-7]. Adequate stability conditions for viscoelastic rods are obtained in this work by means of the second Lyapunov method for a system with an aftereffect.

1. Model of a Viscoelastic Body. Before application of an external load the body is in a natural condition, and at instant of time t = 0 a force is applied to it under whose

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action it deforms. With a uniaxial stressed state stress $\sigma(t)$ is connected with strain $e(t)$ by the relationship

$$\sigma(t) = E\left[e(t) + \int_0^t Q^{\cdot}(t-\tau)e(\tau)\,d\tau\right], \tag{1.1}$$

where E is constant Young's modulus; $Q(t)$ is degree of relaxation; $Q^{\cdot}(t) = dQ(t)/dt$; $Q(0) = 0$. We limit ourselves to studying regular degrees of relaxation for which functions $Q(t)$ are twice continuously differentiable. We assume that for any $t > 0$ the conditions

$$-1 < Q(\infty) < Q(t) < 0; \tag{1.2}$$

$$Q^{\cdot}(t) < 0,\ Q^{\cdot}(\infty) = 0; \tag{1.3}$$

$$Q^{\cdot\cdot}(t) > 0,\ Q^{\cdot\cdot}(\infty) = 0 \tag{1.4}$$

are fulfilled and there exists a constant $T_0 > 0$ such that for any $t \geq 0$

$$Q^{\cdot\cdot}(t) > T_0^{-2}[Q(t) - Q(\infty)]. \tag{1.5}$$

We explain the mechanical idea of these assumptions. We consider the deformation process for a specimen of the form

$$e_1(t) = 0\ (0 \leqslant t \leqslant t_1);\ e_1(t) = e^0 > 0\ (t > t_1). \tag{1.6}$$

From (1.1) and (1.6) we find stress $\sigma_1(t)$ and its derivative with respect to time $\sigma_1^{\cdot}(t)$:

$$\begin{gathered}\sigma_1(t) = 0, \quad \sigma_1^{\cdot}(t) = 0 \quad (0 \leqslant t \leqslant t_1);\\ \sigma_1(t) = E[1 + Q(t-t_1)]e^0, \quad \sigma_1^{\cdot}(t) = EQ^{\cdot}(t-t_1)e^0 \quad (t > t_1).\end{gathered} \tag{1.7}$$

According to (1.3) and (1.7) stress in a specimen decreases with time. It follows from (1.2) that with $t \to \infty$ stress tends towards a positive limiting value independent of the instant of loading t_1 (limiting creep condition [8]). In accordance with (1.4) the relaxation rate $|\sigma_1^{\cdot}(t)|$ decreases monotonically and it tends towards zero with $t \to \infty$. It is noted that conditions (1.2)-(1.4) were formulated in [9, 10]. We assume that $y(t) = Q(t) - Q(\infty)$. It follows from (1.2) and (1.3) that $y(t) > 0$ with $t \geq 0$ and $y(\infty) = 0$. We rewrite relationship (1.5) in the form $y^{\cdot\cdot}(t) > T_0^{-2}y(t)$. We multiply this equality by $y^{\cdot}(t) < 0$ and integrate it from t to infinity. Considering (1.3) we obtain $y^{\cdot}(t) < -T_0^{-1}y(t)$. By integrating this inequality and returning to the original notation we find that

$$0 < Q(t) - Q(\infty) < -Q(\infty)\exp(-t/T_0). \tag{1.8}$$

According to (1.8) condition (1.5) means that with $t \to \infty$ the degree of relaxation tends towards its limiting value more rapidly with the exponent than with characteristic time T_0.

We introduce dimensionless time $\tau = t/T_0$. We assume that $Q_0(\tau) = Q(T_0\tau)$. On the basis of (1.5)

$$Q_0^{\cdot\cdot}(\tau) > Q_0(\tau) - Q_0(\infty) \quad (\tau \geqslant 0). \tag{1.9}$$

In future we require an estimate of the function

$$R(t) = -Q_0^{\cdot}(t) + \int_0^t [Q_0(t-\tau) - Q_0(\infty)]\,d\tau.$$

In view of (1.9) the derivative of function $R(t)$ is negative and $R(t) > R(\infty)$ for any $t \geq 0$. According to (1.3),

$$R(t) \geqslant \lim_{\tau} \int_0^{\tau} [Q_0(s) - Q_0(\infty)]\,ds \ (\tau \to \infty).$$

Integrating by parts and considering (1.8) we obtain

$$R(t)\geqslant\int_0^\infty |Q_0^{\cdot}(s)|\,s\,ds \quad (t\geqslant 0). \tag{1.10}$$

2. Statement of the Problem of Rod Stability. We consider a rectilinear rod of length l made of viscoelastic material. The rod cross section has two axes of symmetry and the center of gravity of the cross section lies on the longitudinal axis. We introduce axis x directed along the longitudinal axis of the rod in the undeformed condition. We designate in terms of ρ material density, S the cross-sectional area, I the moment of inertia of the cross section in relation to the longitudinal axis. Values of ρ, S, and I are assumed to be constant. At instant of time t = 0 to ends of the rod a compressive force with intensity P is applied and the rod bends in the plane of symmetry. Let $v_1(x)$ be the initial rod deflection, $v_2(x)$ be the initial deflection rate, and u(t, x) be rod deflection at point $x \in [0, l]$ at instant of time $t \geq 0$. We assume that rod deflection is quite small so that it is possible to ignore the value $(u')^2 = (\partial u/\partial x)^2$ compared with unity, and the hypothesis of plane sections is fulfilled. If material behavior obeys equation of state (1.1), then function u satisfies an equation [11]

$$\rho S u^{\cdot\cdot}(t) = -Pu''(t) - EI\left[u^{IV}(t) + \int_0^t Q^{\cdot}(t-\tau)\,u^{IV}(\tau)\,d\tau\right] \tag{2.1}$$

with initial conditions

$$u(0) = v_1,\ u^{\cdot}(0) = v_2. \tag{2.2}$$

Here and subsequently in order to reduce the writing we omit argument x. At the rod ends one of a group of conditions is fulfilled

$$u(t, 0) = u(t, l) = 0,\ u''(t, 0) = u''(t, l) = 0; \tag{2.3}$$

$$u(t, 0) = u(t, l) = 0,\ u'(t, 0) = u'(t, l) = 0; \tag{2.4}$$

$$u(t, 0) = u(t, l) = 0,\ u'(t, 0) = 0,\ u''(t, l) = 0. \tag{2.5}$$

Relationships (2.3) characterize a hinged rod, (2.4) a rod whose ends are rigidly fixed, and (2.5) a rod for which one of the ends is rigidly fixed and the other is hinged.

We assume that $P = P_0 + P_1 w^{\cdot}(t)$, where P_0, P_1 are constant values, w(t) is a standard Wiener process, and $w^{\cdot}(t)$ is white noise. Equation (2.1) with initial conditions (2.2) and one of the boundary conditions (2.3)-(2.5) describes deflection of a viscoelastic rod under the action of a compressive sporadic load, and according to [12] it has a unique general solution if initial conditions v_i pertain to space $\mathring{W}_2^1$ the normal to which it is possible to determine by the equation [13]

$$\|v_i\|^2 = \int_0^l [v_i'(x)]^2\,dx.$$

Determination. A rod is called stable in a root-mean-square if for any $\varepsilon > 0$ there exists $\delta > 0$ such that from the inequality $\|v_1\|^2 + \|v_2\|^2 < \delta$ there follows an estimate $\sup_{t,x} Mu^2(t, x) < \varepsilon$ ($t \geq 0$, $x \in [0, l]$, M is mathematical expectation symbol). The problem consists of finding limits for parameters P_0 and P_1 which guarantee rod stability.

3. Transformation of Fundamental Equations. We designate in terms of v_* the maximum value of initial rod deflection. We introduce dimensionless values and parameters: $x_* = x/l$, $t_* = t/T_0$, $v_{1*}(x_*) = v_1(x)/v_*$, $v_{2*}(x_*) = T_0 v_2(x)/v_*$, $u_{1*}(t_*, x_*) = u(t, x)/v_*$, $u_{2*}(t_*, x_*) = T_0 u^{\cdot}(t, x)/v_*$, $w_*(t_*) = T_0^{1/2} w(t)$, $a = EIT_0^2/(\rho S l^4)$, $P_{0*} = P_0 l^2/(EI)$, $P_{1*} = T_0^{3/2} P_1/(\rho S l^2)$. According to [14] the random process $w_*(t_*)$ is a Wiener process. In the new notations relationships (2.1)-(2.5) take the form (the asterisk is omitted in order to reduce the writing)

$$\begin{aligned} du_1 &= u_2(t)dt,\\ du_2 &= -a\left[u^{IV}(t) + \int_0^t Q_0^{\cdot}(t-\tau)\,u_1^{IV}(\tau)\,d\tau + P_0 u''(t)\right]dt - P_1 u_1''(t)\,dw(t); \end{aligned} \tag{3.1}$$

$$u_1(0) = v_1,\ u_2(0) = v_2; \tag{3.2}$$

$$\begin{aligned} &u_1(t,0) = u_1(t,1) = 0,\ u_1''(t,0) = u_1''(t,1) = 0,\\ &u_1(t,0) = u_1(t,1) = 0,\ u_1'(t,0) = u_1'(t,1) = 0,\\ &u_1(t,0) = u_1(t,1) = 0,\ u_1'(t,0) = u_1''(t,1) = 0. \end{aligned} \tag{3.3}$$

We consider the boundary problem

$$Y^{\mathrm{IV}}(x) + \lambda Y''(x) = 0 \tag{3.4}$$

with one of the boundary conditions (3.3). According to [15] there exists a monotonically increasing sequence of positive characteristic values λ_k and nonzero characteristic functions $\varphi_k(x)$ which satisfy the conditions (δ_{kl} is Kronecker symbol):

$$\int_0^1 \varphi_k'(x)\,\varphi_l'(x)\,dx = \delta_{kl},\ \int_0^1 \varphi_k''(x)\,\varphi_l''(x)dx = \lambda_k\delta_{kl}. \tag{3.5}$$

The sequence $\{\varphi_k(x)\}$ is complete in space $W_2{}^2$ whose elements satisfy boundary conditions (3.3). Therefore functions $u_i(t, x)$, $v_i(x)$ may be presented in the form of a series

$$u_i = \sum_{k=1}^{\infty} z_{ik}(t)\,\varphi_k(x),\ v_i = \sum_{k=1}^{\infty} \zeta_{ik}\varphi_k(x). \tag{3.6}$$

We substitute expressions (3.6) in (3.1) and (3.2). We multiply each of the equalities by $\varphi_n''(x)$ and we integrate with respect to x from 0 to 1. Integrating by parts and considering (3.3)-(3.5) we obtain

$$\begin{aligned} &dz_{1n} = z_{2n}(t)dt,\\ &dz_{2n} = -a\lambda_n^2\left[\left(1 - P_0\lambda_n^{-1}\right)z_{1n}(t) + \int_0^t Q_0^{\cdot}(t-\tau)\,z_{1n}(\tau)\,d\tau\right]dt +\\ &\quad + P_1\lambda_n z_{1n}(t)\,dw(t),\\ &z_{1n}(0) = \zeta_{1n},\ z_{2n}(0) = \zeta_{2n}\ (n = 1, 2, \ldots). \end{aligned} \tag{3.7}$$

From (3.3), (3.5), (3.6) and the Cauchy inequality it follows that for any $t \geq 0$, $x \in [0, 1]$

$$\begin{aligned} &u_1^2(t,x) = \left[\int_0^x u_1'(t,\xi)\,d\xi\right]^2 \leqslant \int_0^1 [u_1'(t,\xi)]^2\,d\xi = \sum_{k=1}^{\infty} z_{1k}^2(t),\\ &\|v_i\|^2 = \sum_{k=1}^{\infty} \zeta_{ik}^2. \end{aligned} \tag{3.8}$$

<u>4. Conditions for Rod Stability.</u> It was shown in [1] that with a deterministic load and a quasistatic deformation process for stability of a viscoelastic rod fulfillment of the following inequality is necessary and sufficient

$$P_0 < \lambda_1[1 + Q_0(\infty)]. \tag{4.1}$$

<u>THEOREM.</u> We assume that condition (4.1) is fulfilled and

$$P_1^2 < a\left\{1 + \left[a\lambda_1^2\left(1 + Q_0(\infty) - P_0\lambda_1^{-1}\right)\right]^{-1}\right\}^{-1}\int_0^{\infty} |Q_0^{\cdot}(s)|\,s\,ds. \tag{4.2}$$

Then a viscoelastic rod is stable in a root-mean-square under the action of a sporadic compressive load.

The maximum period of natural bending vibrations of an elastic rod with Young's modulus $E_0 = E[1 + Q_0(\infty)]$ is determined by the equation $T_1 = 2\pi[\rho S l^4/(E_0 I\lambda_1{}^2)]^{1/2}$. We write in terms of $P_e = E_0 I\lambda_1 l^{-2}$ Euler critical force for an elastic rod. In the original notation conditions for stability of a viscoelastic rod (4.1) and (4.2) take the form

$$P_0/P_e < 1,$$
$$(P_0/P_e)^2 < N(1 - P_0/P_e)[1 + 4\pi^2(T_0/T_1)^2(1 - P_0/P_e)]^{-1},$$
$$N = [1 + Q(\infty)]^{-1}\int\limits_0^\infty |Q^{\cdot}(s)|\, s\, ds. \tag{4.3}$$

As an example we consider a standard viscoelastic material whose behavior is described by the equation $\sigma^{\cdot} + T_0^{-1}\sigma = Ee^{\cdot} + E_0T_0^{-1}e$. Here E and E_0 are instantaneous and long-term elasticity moduli, T_0 is characteristic relaxation time. With $P_0 = 0$ the condition for rod stability takes the form

$$|P_1| < P_e[(E/E_0 - 1)T_0]^{1/2}[1 + 4\pi^2(T_0/T_1)^2]^{-1}.$$

5. Preliminary Estimates. According to (4.1) there exists $\alpha > 0$ such that for any $n \geq 1$

$$\alpha_n = 1 + Q_0(\infty) - P_0\lambda_n^{-1} \geqslant 1 + Q_0(\infty) - P_0\lambda_1^{-1} \geqslant \alpha. \tag{5.1}$$

Functions $\psi(x) = a\, x^2[1 + Q_0(\infty) - P_0x^{-1}]$ increase monotonically with $x \geq P_0[2(1 + Q_0(\infty))]^{-1}$. Thus it also follows from (4.1) that with any $n > 1$

$$\psi_n > \psi_1\ (\psi_n = \psi(\lambda_n)). \tag{5.2}$$

On the basis of (4.2) there exists $\beta > 0$ such that

$$\int\limits_0^\infty |Q_0^{\cdot}(s)|\, s\, ds - P_1^2a^{-1}(1 + \psi_1^{-1}) \geqslant \beta. \tag{5.3}$$

It follows from (1.10), (5.2), and (5.3) that

$$R(t) - P_1^2a^{-1}(1 + \psi_1^{-1}) \geqslant \beta. \tag{5.4}$$

6. Proof of the Theorem. We calculate the differential of functional

$$W_{1n}(t) = z_{2n}^2(t) + a\lambda_n^2\Big[\big(1 + Q_0(t) - P_0\lambda_n^{-1}\big)z_{1n}^2(t) - \\ - \int\limits_0^t Q_0^{\cdot}(t-\tau)(z_{1n}(t) - z_{1n}(\tau))^2\, d\tau\Big]. \tag{6.1}$$

From the Ito equation and (3.7) we obtain

$$dW_{1n} = -a\lambda_n^2\Big[\int\limits_0^t Q_0^{\cdot\cdot}(t-\tau)(z_{1n}(t) - z_{1n}(\tau))^2\, d\tau - \\ - \big(Q_0^{\cdot}(t) + P_1^2a^{-1}\big)z_{1n}^2(t)\Big]dt + 2P_1\lambda_nz_{1n}(t)\, z_{2n}(t)\, dw(t). \tag{6.2}$$

We find the differential of the functional

$$W_{2n}(t) = z_{2n}(t) + a\lambda_n^2\int\limits_0^t [Q_0(t-\tau) - Q_0(\infty)]\, z_{1n}(\tau)\, d\tau. \tag{6.3}$$

Following from (3.7) we have

$$dW_{2n} = -a\lambda_n^2\alpha_nz_{1n}(t)\, dt + P_1\lambda_nz_{1n}(t)\, dw(t). \tag{6.4}$$

From relationships (3.7), (6.2)-(6.4) it follows that the differential of the functional

$$W_{3n}(t) = W_{2n}^2(t) + \psi_n\big[z_{1n}^2(t) + W_{1n}(t)\big] \tag{6.5}$$

equals

$$dW_{3n} = -a\lambda_n^2\psi_n\Big[-\big(Q_0^{\cdot}(t) + P_1^2 a^{-1}\big(1+\psi_n^{-1}\big)\big)z_{1n}^2(t) + \int_0^t Q_0^{\cdot\cdot}(t-\tau)(z_{1n}(t)-z_{1n}(\tau))^2 d\tau + 2z_{1n}(t)\int_0^t (Q_0(t-\tau) - Q_0(\infty))z_{1n}(\tau)d\tau\Big]dt + 2P_1\lambda_n z_{1n}(t)[W_{2n}(t) + \psi_n z_{2n}(t)]\,dw(t). \tag{6.6}$$

We transform expressions in the right-hand part of (6.6):

$$2z_{1n}(t)\int_0^t (Q_0(t-\tau) - Q_0(\infty))z_{1n}(\tau)d\tau = -\int_0^t (Q_0(t-\tau) - Q_0(\infty))(z_{1n}(t) - z_{1n}(\tau))^2 d\tau + z_{1n}^2(t)\int_0^t (Q_0(t-\tau) - Q_0(\infty))d\tau + \int_0^t (Q_0(t-\tau) - Q_0(\infty))z_{1n}^2(\tau)d\tau. \tag{6.7}$$

It follows from relationships (6.6) and (6.7) that

$$dW_{3n} = -a\lambda_n^2\psi_n\Bigg[\int_0^t Q_0^{\cdot\cdot}(t-\tau) - (Q_0(t-\tau) - Q_0(\infty))(z_{1n}(t)-z_{1n}(\tau))^2 d\tau + \big(R(t) - P_1^2 a^{-1}\big(1+\psi_n^{-1}\big)\big)z_{1n}^2(t) + \int_0^t (Q_0(t-\tau) - Q_0(\infty))z_{1n}^2(\tau)d\tau\Bigg]dt + 2P_1\lambda_n z_{1n}(t)(W_{2n}(t) + \psi_n z_{2n}(t))\ dw(t). \tag{6.8}$$

We integrate relationship (6.8) from 0 to t and we work out the mathematical expectation for both parts of the equality obtained. Considering (1.9), (5.1), and (5.4) and conditions (1.2)-(1.4) we find that $MW_{3n}(t) \le MW_{3n}(0)$. By substituting in this relationship expressions (6.1), (6.3), and (6.5) and amplifying the inequality we have

$$I_n(t) = M\Bigg\{\Bigg[z_{2n}(t) + a\lambda_n^2\int_0^t (Q_0(t-\tau) - Q_0(\infty))z_{1n}(\tau)d\tau\Bigg]^2 + \psi_n\Bigg[z_{1n}^2(t) + z_{2n}^2(t) + a\lambda_n^2\Bigg(\big(1 + Q_0(t) - P_0\lambda_n^{-1}\big)z_{1n}^2(t) - \int_0^t Q_0^{\cdot}(t-\tau)(z_{1n}(t) - z_{1n}(\tau))^2 d\tau\Bigg)\Bigg]\Bigg\} \leqslant a\lambda_n^2\big(1 + a\lambda_n^2\big)\zeta_{1n}^2 + \big(1 + a\lambda_n^2\big)\zeta_{2n}^2. \tag{6.9}$$

From relationships (5.1), (5.2), and (6.9) and conditions (1.2)-(1.4) it follows that $I_n \ge \alpha\ \lambda_n^2(1 + \alpha a\lambda_n^2)Mz_{1n}^2(t)$. From this relationship and (6.9) it follows that there exists a constant $c_3 > 0$ independent of n such that with $t \ge 0$

$$Mz_{1n}^2(t) \leqslant c_3\big(\zeta_{1n}^2 + \zeta_{2n}^2\big). \tag{6.10}$$

We sum equality (6.10) with respect to n. Considering (3.8) we obtain

$$Mu_1^2(t,x) \leqslant c_3(\|v_1\|^2 + \|v_2\|^2) \quad (t \geqslant 0,\ x \in [0,1]). \tag{6.11}$$

Proof of the theorem follows from inequality (6.11).

7. Instability of an Elastic Rod with a Sporadic Compressive Load. We consider bending of an elastic rod compressed over the ends by a force P, $P_0 < P_e$. We take the dimensional initial perturbation in the form $v_1 = 0$, $v_2 = \delta\varphi_1(x)$ (δ = const). Here dimensionless values $u_i(t, x)$ are determined by the equations

$$u_1 = \delta z_1(t)\varphi_1(x),\ u_2 = \delta z_2(t)\varphi_1(x). \tag{7.1}$$

Coefficients $z_1(t)$ and $z_2(t)$ satisfy the set of equations

$$dz_1 = z_2\,dt,\ dz_2 = -a\lambda_1^2(1 - P_0\lambda_1^{-1})\,z_1\,dt + P_1\lambda_1 z_1\,dw(t),$$
$$z_1(0) = 0,\ z_2(0) = 1. \tag{7.2}$$

It follows from (7.2) and the Ito equation that functions $X_1 = Mz_1^2(t)$, $X_2 = Mz_1(t) \times z_2(t)$, $X_3 = Mz_2^2(t)$ are solutions of the set of equations ($\psi_1 = a\,\lambda_1^2(1 - P_0\lambda_1^{-1})$, $\theta_1 = P_1^2\lambda_1^2$)

$$X_1^{\cdot} = 2X_2,\ X_2^{\cdot} = -\psi_1 X_1 + X_3,\ X_3^{\cdot} = \theta_1 X_1 - 2\psi_1 X_2,$$
$$X_1(0) = 0,\ X_2(0) = 0,\ X_3(0) = 1. \tag{7.3}$$

We write a characteristic equation for system (7.3):

$$f(k) = 0,\ f(x) = x^3 + 4\psi_1 x - 2\theta_1. \tag{7.4}$$

Function f(x) increases monotonically, $f(0) = -2\theta_1$, $f(\infty) = \infty$. This means that with $P_1 \neq 0$ for Eq. (7.4) the sole real positive root $k_1 = \kappa$. Two other roots are determined by the equations $k_{2,3} = (-\kappa \pm i\omega)/2$, $\omega^2 = 3\kappa^2 + 16\psi_1 > 0$. We write the solution of set of Eqs. (7.3) in the form

$$\begin{aligned}
X_1 &= -[\varkappa(A\cos\omega t/2 - B\sin\omega t/2) + \omega(A\sin\omega t/2 + \\
&\quad + B\cos\omega t/2)]\exp(-\varkappa t/2) + 2C\varkappa\exp(\varkappa t),\\
X_2 &= (1/4)[(\varkappa^2 - \omega^2)(A\cos\omega t/2 - B\sin\omega t/2) + 2\varkappa\omega(A\sin\omega t/2 + \\
&\quad + B\cos\omega t/2)]\exp(-\varkappa t/2) + C\varkappa^2\exp(\varkappa t),\\
X_3 &= [(2\theta_1 + \psi_1\varkappa)(A\cos\omega t/2 - B\sin\omega t/2) + \psi_1\omega(A\sin\omega t/2 + \\
&\quad + B\cos\omega t/2)]\exp(-\varkappa t/2) + 2(\theta_1 - \psi_1\varkappa)C\exp(\varkappa t).
\end{aligned} \tag{7.5}$$

Constants A, B, C have the form $A = 4\kappa^2[\theta_1(9\kappa^2 + \omega^2)]^{-1}$, $B = \kappa(\omega^2 - 3\kappa^2)[\theta_1\omega(9\kappa^2 + \omega^2)]^{-1}$, $C = (\kappa^2 + \omega^2)[2\theta_1(9\kappa^2 + \omega^2)]^{-1}$.

It follows from relationships (7.1) and (7.5) that for any $\delta > 0$ the value $Mu_1^2(t, x)$ tends towards infinity with $t \to \infty$. Consequently, an elastic rod in a root-mean-square is unstable with a sporadic component of load of the white noise type of arbitrary intensity.

According to (4.1) with a deterministic longitudinal load material toughness leads to a reduction in critical force. As the example provided indicates, with a sporadic longitudinal load the presence of toughness plays a positive role: with fulfillment of inequality (4.3) a viscoelastic rod is stable, but an elastic rod is unstable.

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SHELL THEORY BASED ON INVARIANTS

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The precise theory is considered for finite strains of a three-dimensional body subordinate to the hypothesis of holding a normal element against a reference (central) surface. The first and second invariants of the strain tensor for a Green surface parallel to the reference surface are used as a measure of physical strains. It is shown that from the invariants of physical strains it is possible to determine any invariant characteristic of an elastic body: energy, stress tensor invariants, stress intensity, etc. A general definition is given for strain invariants of an arbitrary surface as components of the relative change in the square of a surface element. There is simplification of invariants with small strains and any distortions of thin shells. Expressions are obtained for the change in coefficients of the first and second quadratic forms of the central surface for small strains, and arbitrary and small displacements.

1. Geometry of a Three-Dimensional Body. We assume that $\mathbf{R}$ is radius vector of a three-dimensional body in the undeformed condition which is expressed in terms of reference surface radius vector $\mathbf{r}$ and the unit vector of the normal to the surface in the form $\mathbf{R} = \mathbf{r} + z\mathbf{n}$. In the general case $\mathbf{r}$ will be assumed to be independent of arbitrary curvilinear coordinates α_i. Coefficients of the first invariant form of the reference surface $a_{ij} = \mathbf{r}_{,i}\mathbf{r}_{,j}$, and for the surface z = const $A_{ij} = \mathbf{R}_{,i}\mathbf{R}_{,j}$. Here and subsequently i, j = 1, 2: indices after a comma signify differentiation with respect to α_i. The vector of the normal to surface z = const coincides with the vector of the normal to the base: $\mathbf{n} = (\mathbf{r}_{,1} \times \mathbf{r}_{,2})\, d_{aa}^{-1/2}$. For further convenience we adopt the following definition of the value $d_{\beta\gamma}$ which depends on the coefficients of any two quadratic forms β_{ij}, γ_{ij} ($d_{\beta\gamma} \neq d_{\gamma\beta}$):

$$d_{\beta\gamma} = \det\begin{Vmatrix}\beta_{11} & \beta_{12}\\ \gamma_{21} & \gamma_{22}\end{Vmatrix} = \beta_{11}\gamma_{22} - \beta_{12}\gamma_{21}.$$

Then $d_{aa} = a_{11}a_{22} - a_{12}^2$ is discriminant of quadratic form $a_{ij}d\alpha_i d\alpha_j$. The square of an element of area dF^2 of surface z = const has the form $dF^2 = d_{AA}d\alpha_1^2 d\alpha_2^2$. We assume that deformation of a three-dimensional body follows the hypothesis of holding a normal element against a reference surface [1]. In the deformed condition $\mathbf{R}^\vee = \mathbf{r}^\vee + z\mathbf{n}^\vee$, $a_{ij}^\vee = \mathbf{r}_{,i}^\vee \mathbf{r}_{,j}^\vee$, $A_{ij}^\vee = \mathbf{R}_{,i}^\vee \cdot \mathbf{R}_{,j}^\vee$, $\mathbf{n}^\vee = (\mathbf{r}_{,1}^\vee \times \mathbf{r}_{,2}^\vee)\, d_{aa}^{\vee -1/2}$, $dF^{\vee 2} = d_{AA}^\vee d\alpha_1^2 d\alpha_2^2$. Here for $d_{\beta\beta}$, where $\beta_{ij} = \gamma_{ij}^\vee$, we adopt the symbol $d_{\gamma\gamma}^\vee$.

2. Determination of Physical Strain Invariants. We consider surface z = const in the deformed condition. Assuming $A_{ij}^\vee = A_{ij} + 2E_{ij}$ and formulating the ratio $dF^{\vee 2}/dF^2$, we obtain

$$dF^{\vee 2}/dF^2 = 1 + 2I_E + 4I_{EE}; \tag{2.1}$$

$$I_E = (d_{AE} + d_{EA})/d_{AA}; \tag{2.2}$$

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